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Georgia Institute of Technology

School of Civil and Environmental Engineering

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MEMORANDUM

To: Professor Takeru Igusa
Department of Civil Engineering
Johns Hopkins University
FAX 410-516-7473

From: Bruce R. Ellingwood
Professor

Re: Reliability-based simulation for performance-based design (Award No. 8012-48306)

Per my previous note, attached is copy of the final report on this project.

Attachment:
Nie and Ellingwood, "New developments in directional methods for system reliability assessment"

Cc: JWBerkowitz
Office of Sponsored Programs
Georgia Tech Research Corporation

School of Civil and Environmental Engineering
Atlanta, Georgia 30332-0355 U.S.A.
PHONE 404•894•2201
FAX 404•894•2278

New Developments in Directional Methods for System Reliability Assessment

Jinsuo Nie
Department of Civil Engineering, The Johns Hopkins University, Baltimore, MD

Bruce R. Ellingwood
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA

Keywords: Computational mechanics; Directional simulation; Reliability; Statistics.

ABSTRACT: Accurate estimates of structural system reliability are important in developing performance-based design codes. The only feasible approaches for assessment of realistic systems involve various simulation techniques involving nonlinear finite element analysis. However, simulations are prone to large variation and large computational demand when finite-element analysis is involved. Directional simulation, an efficient simulation technique, is improved by the use of deterministic point sets which have high fidelity in representing the joint probability distribution with a limited number of points. An error measure is introduced to estimate and compare the error levels of different point sets. Fekete point sets are shown to be an optimal choice. Efficient procedures to generate Fekete points are proposed. The method is illustrated through a reliability analysis of a portal frame.

1 INTRODUCTION

System reliability analysis is complicated by many factors, including the dimensionality of the random variable space, the inherently small failure probability, highly nonlinear limit states, the possibility of multiple design points and difficulty in identifying all dominant failure modes. Therefore, the system reliability analysis required is well beyond the ability of closed-form analytical solutions, direct numerical integration or classical failure-mode/path approaches. The First Order / Second Order Reliability Methods (FORM/SORM) may not deal effectively with highly nonlinear limit states and multiple design points. To obtain acceptable accuracy for a broad class of realistic structures, simulation techniques appear to be the final resort. However, when the target failure probability is small, which is common in structural system reliability, the required number of samples in a naive Monte Carlo simulation to achieve an acceptable level of accuracy can be very large, and the standard error in the estimate of probability of failure may be high.

To this end, various variance reduction techniques have been applied to reduce the required number of samples; these include stratified sampling, Latin Hypercube sampling, importance sampling and directional simulation. Directional simulation is recognized as efficient [Deák, 1980; Ditlevsen et al., 1990; Kijawatworawet et al., 1998; Melchers, 1990, 1994; Bjerager, 1988, among others]. Nevertheless directional simulation still leaves room for further improvement in the manner in which the distributions are randomly sampled. The essential reason of the low efficiency of all the sampling techniques is that the sample points (directions) representing the random variables cannot represent their underlying distributions very well for a small number of points.

To this end, deterministic point sets provide a technique to overcome the problem of low representativity. These sets are so named to distinguish them from sets that are randomly generated, and have been developed in different domains of science and engineering. The categories of deterministic point sets include spherical t-designs [Hardin and Sloane, 1996], GLP point sets [Fang and Wang, 1994], Fekete point sets [Nie and Ellingwood, 2000], spiral point sets [Saff and Kuijlaars, 1997] and AHDM point sets [Katsuki and Frangopol, 1994, 1998]. As a new addition to direction simulation, deterministic directional methods share the same general solution strategy with directional simulation, except that the directions (points on the unit hypersphere) are not generated randomly. The basic procedure is to evaluate the failure probability utilizing the χ^2 distribution in the standard normal space transformed from the original distribution [Nie and Ellingwood, 2000; Katsuki and Frangopol, 1994].

Deterministic point sets from the uniform distribution are emphasized herein, since the uniform distribution is commonly used as a starting point in directional simulation. Previous research has indicated that so-called Fekete point sets, which minimize potential energy of the points on the unit hypersphere, have particularly attractive features in terms of accuracy. Because generating a large number of Fekete points is time-consuming, the basic idea is to store the points permanently, and use them repeatedly when possible. The overall effort of simulation-based system reliability assessment can thus be reduced.

2 DETERMINISTIC POINT SETS

Spherical t-designs

Spherical t-design theory is the basis for assessing accuracy of other point sets. A spherical t-design [Hardin and Sloane, 1996] is a technique for spherical quadrature. It identifies a point set $\mathcal{P} = [\vec{x}_1, \ldots, \vec{x}_m]$ on the unit hypersphere $\mathcal{S}^{n-1}$, where n is the dimension of the space, such that the following equally-weighted quadrature rule,

$$\int_{\mathcal{S}^{n-1}} p(\vec{x}) \frac{dS}{V_{\mathcal{S}^{n-1}}} = \frac{1}{m} \sum_{i=1}^{m} p(\vec{x}_i) \tag{1}$$

is satisfied for all polynomials $p(\vec{x})$ of degree $\leq t$. It should be emphasized here that the above quadrature is exact iff $\mathcal{P}$ is a spherical t-design. It works in a way analogous to Gauss Quadrature, since both generate the integration points and store them for use in a broad spectrum of problems.

In order to form a spherical t-design, the minimum number of points (cardinality) in $\mathcal{P}$ is required as a lower bound [Conway and Sloane, 1998; Nie and Ellingwood, 2000]:

$$M_t = \begin{cases} 2\begin{pmatrix} \frac{t-1}{2}+n-1 \\ n-1 \end{pmatrix}, & \text{if } t \text{ is odd} \\ \begin{pmatrix} \frac{t}{2}+n-1 \\ n-1 \end{pmatrix} + \begin{pmatrix} \frac{t}{2}+n-2 \\ n-1 \end{pmatrix}, & \text{if } t \text{ is even,} \end{cases} \tag{2}$$

where () is the binomial coefficient. A spherical t-design $\mathcal{P}$ is denoted a tight spherical t-design if its cardinality equals M_t. A literature review indicated that tight spherical t-designs exist only for some small n and t. M_t provides guidance in selecting the cardinalities for other sets.

GLP Point Sets

The Good Lattice Point (GLP) is one of a number of Number-Theoretical point sets on the hypercube introduced by Fang and Wang [1994] for use in numerical analysis, and has been shown to be the most evenly scattered set on the hypercube among the Number-Theoretical point sets. A GLP point set can be generated from an integral generating vector $\vec{h}_n = (m; h_1, \ldots, h_n)$, following the notation in Fang and Wang [1994], where m is the cardinality of the GLP point set, $1 \leq h_i \leq m$ and $h_i \neq h_j,\ for\, i \neq j$, and $n < m$, and the greatest common divisors $(m, h_i) = 1$. Then a point set $\mathcal{P} = \{\vec{x}_1, \ldots, \vec{x}_m\}$ is called a Lattice Point set if

$$\vec{x}_i = \left[\frac{2ih_1 - 1}{2m}, \left\{\frac{2ih_2 - 1}{2m}\right\}, \ldots, \left\{\frac{2ih_n - 1}{2m}\right\}\right], \quad i = 1, \ldots m, \tag{3}$$

where $\{\}$ is the operator to return the fractional part of a real number. If the point set $\mathcal{P}$ has the lowest discrepancy, a measure describing the uniformity, among all possible generating vectors, $\mathcal{P}$ is called a GLP point set. The generating vectors are found by searching a reduced space of those possible choices of $\vec{h}_n$. GLP point sets can be transformed from hypercube to hypersphere efficiently, and vice versa [Fang and Wang, 1994; Nie, 2002].

Fekete Point Sets

The equally weighted formulation in Equation 1 suggests that the distribution of a spherical t-design should be uniform. Fekete Point Sets are used to approximate spherical t-designs when they are difficult to determine. Fekete points achieve uniform distribution [Saff and Kuijlaars, 1997; Nie and Ellingwood, 2000] by minimizing the potential energy (PE) in a set of points on the sphere if the points are considered particles with unit charge.

Let $\mathcal{P} = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a point set on the unit sphere, with each point having a unit charge. The potential energy is defined as,

$$PE(1, \mathcal{P}) = \sum_{1 \leq j < k \leq m} \frac{1}{|\vec{x}_j - \vec{x}_k|}. \tag{4}$$

Then Fekete points $\mathcal{P}^*$ are the set minimizing $PE(1, \mathcal{P})$ [Saff and Kuijlaars, 1997]. Fekete points were originally defined for $\mathcal{S}^2$; however this concept can be generalized to a higher dimensional space.

A Fekete point set can be obtained by simulating a physical system of particles restrained on the unit hypersphere. Besides the pattern search method introduced in Nie and Ellingwood [2000], two faster generating methods are proposed in Nie [2002]. One is a revised pattern search, and the second uses a different PE expression,

$$PE(2, \mathcal{P}) = \sum_{1 \leq j < k \leq m} \frac{1}{|\vec{x}_j - \vec{x}_k|^2}. \tag{5}$$

An iterative updating scheme has been derived for minimizing $PE(2, \mathcal{P})$. The time bound for all three procedures have the same time bound of $O(m^2)$ for any single iteration.

Figure 1 compares visually in 3D various point sets with comparable numbers of points. The spherical t-design, Fekete points, GLP points and spiral points are more evenly scattered than AHDM or random sets.

3 ERROR ESTIMATES

Deterministic point sets must be subjected to uniformity tests before they are applied to reliability computation or numerical integration. In a number of studies [Nie and Ellingwood, 2000; Katsuki and Frangopol, 1994, 1998, among others], hyperplanes with fixed orientation and distance to the origin have been used to test the accuracy and efficiency of point sets. One potential problem with such tests is that some directions may be preferred because only part of the point set contributes significantly. In order to test all directions (points), a Hyperplane Test uses a randomly oriented hyperplane and repeats the experiment for a sufficiently large number of times. The distance of the hyperplane to the origin is set to 3, which yields a failure probability of 1.34990×10^{-3} for any realization of the hyperplane. 10,000 randomly generated hyperplanes were used to get the error statistics in the following results.

Figure 2 shows the Hyperplane Test results in 3D; Table 1 shows some results in higher dimensional space. The point sets in Table 1 are denoted as "Cn-m-t", where n is the dimension of the problem, m is the cardinality of a point set, t is such that m satisfies the lower bound in Equation 2 and C is the class of a point set as defined in the caption to Figure 2. The basis for the spiral points and AHDM methods is discussed elsewhere [Nie and Ellingwood, 2000; Katsuki and Frangopol, 1998].

Table 1: Hyperplane Tests for High Dimensions

	Error (%)		
Point Set	Mean	Std	Max.
F4-135-11	0.902	0.685	4.954
G4-135-11	3.732	3.472	20.48
R4-135-11	31.59	22.43	115.8
F7-2129-13	0.600	0.454	3.058
G7-2129-13	2.970	2.455	23.66
R7-2129-13	12.22	9.196	63.03
F9-3997-11	1.092	0.823	6.344
G9-3997-11	5.663	4.650	34.78
R9-3997-11	11.40	8.752	67.51
F12-4661-9	2.553	1.942	12.66
G12-4661-9	8.698	6.866	47.22
R12-4661-9	12.67	9.695	82.80
F16-70864-12	0.926	0.692	4.591
G16-70864-12	5.773	5.099	47.90
R16-70864-12	3.798	2.887	20.14
F19-70864-11	1.443	1.090	7.251
G19-70864-11	4.394	3.782	30.17
R19-70864-11	4.088	3.105	21.49
F20-100k-11	0.888	0.671	4.443
R20-100k-11	3.439	2.591	17.55

The general order in 3D in terms of increasing error is spherical t-design, Fekete points, GLP points, spiral points, AHDM points and random points. Table 1 suggests that the Fekete point sets generally result in smaller errors than other sets for problems of comparable size [Nie, 2002].

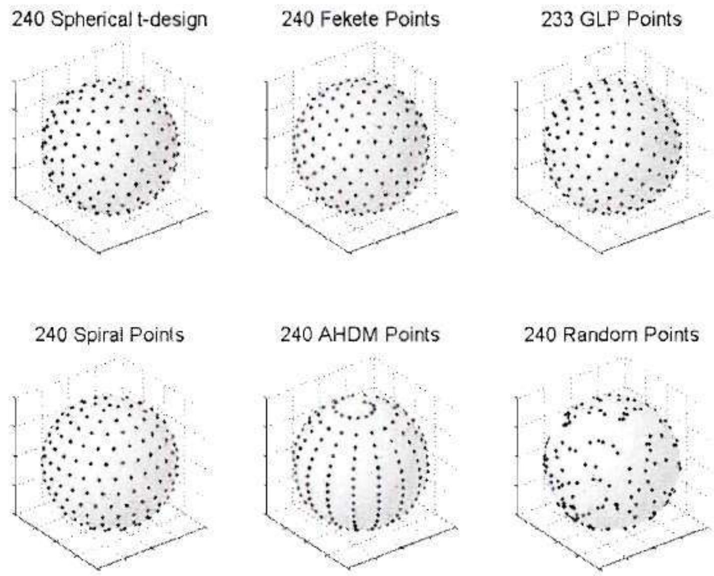

Figure 1: A Visual Comparison of Different Point Sets

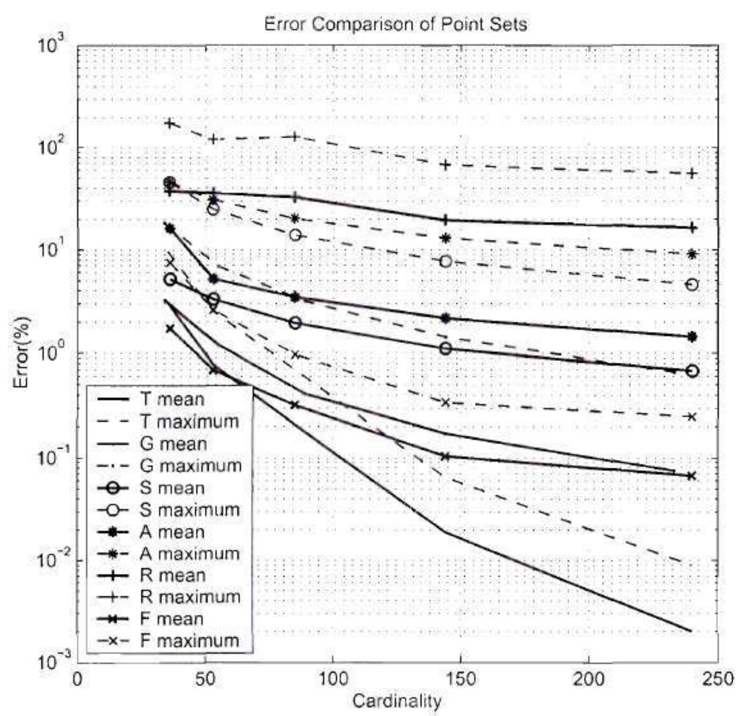

Figure 2: Error Statistics in 3D (T-Spherical t-design; G-GLP Points; S-Spiral Points; A-AHDM Points; R-Random Points; F-Fekete Points)

4 A PORTAL FRAME EXAMPLE

The Fekete point method can be coupled to a neural network technique to focus the simulation effort in significant regions of the probability space. Details are given in Nie [2002]. This coupled method was applied to the portal frame in Figure 3, first examined by Ditlevsen et al. [1990] and later by Nie and Ellingwood [2000]. This frame is modeled

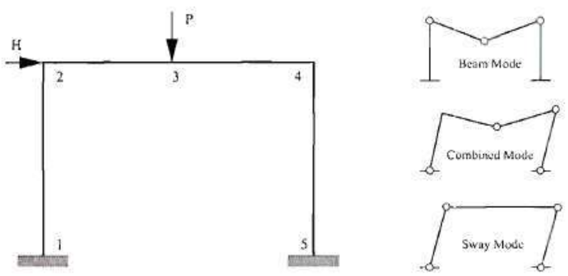

Figure 3: Rigid-plastic Portal Frame

as a rigid-plastic structure subjected to concentrated gravity load P at the middle of the beam, and horizontal load H at the top of the column. Loads and geometry properties of the structure are considered deterministic, while the yield moments M_i, $i = 1, \ldots, 5$ are modeled as lognormal random variables which are independently and identically distributed. After the random variables are transformed into standard normal space, the three collapse modes can be expressed as,

$$
\begin{aligned}
G_B &= e^{\xi z_2+\lambda} + 2e^{\xi z_3+\lambda} + e^{\xi z_4+\lambda} - 1.15 \qquad (6) \\
G_S &= e^{\xi z_1+\lambda} + e^{\xi z_2+\lambda} + e^{\xi z_4+\lambda} + e^{\xi z_5+\lambda} - 2.40 \\
G_C &= e^{\xi z_1+\lambda} + 2e^{\xi z_3+\lambda} + 2e^{\xi z_4+\lambda} + e^{\xi z_5+\lambda} - 3.55,
\end{aligned}
$$

where $\xi = 0.2462$ and $\lambda = -0.03031$. The failure of the system is the union of the three individual collapse modes.

The "exact" failure probability 5.45191×10^{-5} was found by directional simulation with 50,000 directions. The sampling error was estimated as 1.79×10^{-6} and the COV was 3.28% [Nie and Ellingwood, 2000]. Fekete sets F5-196-10 and F5-2080-21 were used for training and simulating two types of neural networks; consequently only 482 and 403 points (directions) were used respectively. The estimated failure probabilities by these two neural networks were 5.43×10^{-5} and 5.45×10^{-5}; the associated errors are 0.44% and 0.06%. This coupled method leads to a computational saving of approximately 99.04% and 99.19% over naive directional simulation respectively.

5 CONCLUSION

In some cases, generating a deterministic point set may be time-consuming. However, when the number of points selected is based on the guidelines of error estimates and the point set has been tested accordingly, those points can be stored and reused. Despite the higher front-end cost of generating the points, their use can reduce the overall cost of system reliability analysis using finite element analysis and simulation, as fewer FE calls are required to estimate structural reliability to a given degree of accuracy.

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